Indian Statistical Institute,Bangalore Centre

B.Math.(Hons.)II Year-2013-14,First Semester

Optimization

Supplementary Exam — 31 Dec 2013, 10am - 1pm.

Instructor: P.S.Datti — Max.Marks: 50

NOTE: Solve all the questions. WRITE NEATLY.

1. Suppose A is a real symmetric $n \times n (n > 1)$ matrix with principal minors denoted by Δ_k, $1 \leq k \leq n$. If $\Delta_k > 0$ for $k < n$ and $\Delta_n \geq 0$, show, by induction, that A has a decomposition of the form $A = LL^t$, where L is a lower triangular matrix. **(5)**

2. Formulate the dual problem of the following primal problem:

$$\text{minimize} \quad 2x_1 + x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 3 \\ 2x_1 + x_2 + 3x_3 &= 5 \end{aligned}$$

$$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0$$

(2)

3. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Are the columns of A linearly independent? If so, find orthonormal vectors $u^{(1)}, u^{(2)}, u^{(3)}$ in $\mathbb{R}^4$ such that the subspace spanned by them is the subspace spanned by the columns of A. **(5)**

4. Suppose $u^{(1)}, u^{(2)}, \ldots, u^{(k)}$ are orthonormal vectors in $\mathbb{R}^n$. For any vector $x \in \mathbb{R}^n$, show that $\|x - \sum_{j=0}^{k}(x, u^{(j)})u^{(j)}\| \geq \|x - \sum_{j=0}^{k} \lambda_j u^{(j)}\|$, for any real numbers λ_j with equality iff $(x, u^{(j)}) = \lambda_j$ for all j. **(5)**

5. Let $s > 0$ and $t > 0$ with $s+t = 1$. Consider the matrix $A = \begin{pmatrix} s & t & 0 \\ 0 & s & t \\ t & 0 & s \end{pmatrix}$. State clearly the reasons why the limit $\lim_{k \to \infty} A^k$ exists and then find the limit. **(2+2)**

6. Suppose A is a real $m \times n$ matrix and λ is a non-zero eigenvalue of A^tA of algebraic multiplicity k. Show that λ is also an eigenvalue of AA^t of algebraic multiplicity k. **(6)**

7. Let $y_0 = 1$ and $z_0 = 3$. Define for $k = 0, 1, 2, \ldots$

$$\begin{aligned} y_{k+1} &= 0.6y_k + 0.5z_k \\ z_{k+1} &= 0.4y_k + 0.5z_k \end{aligned}$$

Find the limits of y_k and z_k as $k \to \infty$. **(4)**

8. Find a solution of the following LPP using simplex method:

$$\text{minimize} \quad 5x_1 - 8x_2 - 3x_3$$

subject to

$$\begin{aligned} 2x_1 + 5x_2 - x_3 &\leq 1 \\ -3x_1 - 8x_2 + 2x_3 &\leq 4 \\ -2x_1 - 12x_2 + 3x_3 &\leq 9 \end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

(5)

9. Given that the following LPP:

$$\text{maximize} \quad \frac{3}{4}x_1 - 20x_2 + \frac{1}{2}x_3 - 6x_4$$

subject to

$$\begin{aligned} \frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 &\leq 0 \\ \frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 &\leq 0 \\ x_3 &\leq 1 \end{aligned}$$

$$x_i \geq 0,\ i = 1, 2, 3, 4.$$

has an optimal solution $(1, 0, 1, 0)$. Write down the corresponding dual problem and solve it by simplex method. Also verify the duality theorem for this problem. **(2+4)**

10. (a) Without using the simplex method, minimize the objective function $5x_1 + 3x_2 + 4x_3$ subject to the constraints $x_1 + x_2 + x_3 \geq 1$, $x_i \geq 0$, $i = 1, 2, 3$. **(2)**

    (b) What is the dual problem and what is its solution? **(2+2)**

11. Using the theorem of the alternative, show that the following system

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

does not have a solution. **(2)**